# Black hole entropy and signal communication: a mathematical perspective

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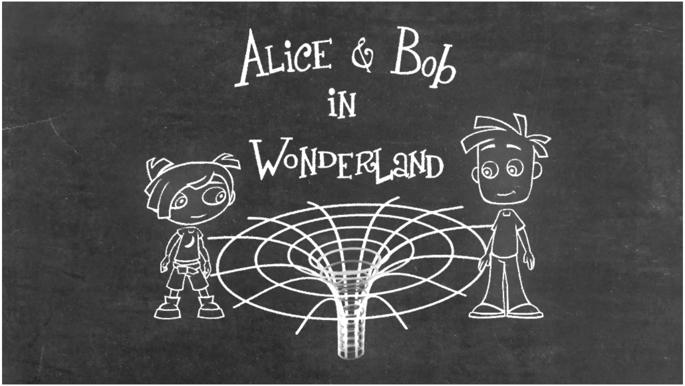
# Quantum information

Quantum Information is a fast-growing subject. A primary problem in the subject is to describe the information carried by the quantum channel. The mathematical research activity on Quantum Information has so far been mainly focused on finite-dimensional systems, and it is conceptually remarkably rich.

QFT recently prompted to study the entropy content of the theory. In QFT, particle creation/annihilation occurs, so the system deals with infinite degrees of freedom. Therefore *Quantum Information for systems with infinitely many degrees of freedom is needed*. The root of this analysis can be traced back to the famous Bekenstein-Hawking geometric interpretation of black hole entropy.




Cartoon made by the Perimeter Institute

Alice, Bob and a black hole

# Black hole thermodynamics (70's)

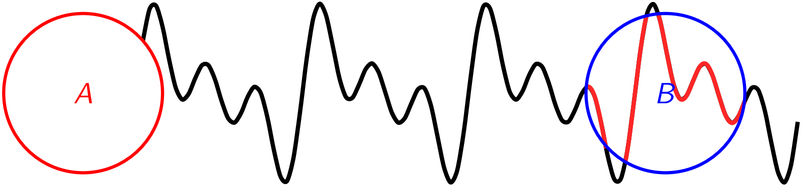
*Schematically*: $B$ be a black hole (for example, Schwarzschild black hole)

- the event horizon area $A$ of $B$ cannot decrease, $dA \geq 0$ (Hawking et al.)
- $S = A/4$ with $S$ the black hole entropy (Bekenstein)
- a (quantum) black hole $B$ emits a thermal radiation with temperature $T = \kappa/2\pi$ (Hawking), $\kappa =$ surface gravity
- the generalized laws of thermodynamics hold for $B$ (Carter, Hawking, Bardeen)


Figure: Hokusai. *The Great Wave*

# The information carried by a classical wave

*Suppose that Alice encodes and sends information by an undulatory signal, what information can Bob get by the wave packet in a given region at later time?*




Bob has access only to the portion of the wave that is in his lab at a given time. We are interested in the local information or information density of the wave packet.

# Wave packets

By a Klein-Gordon wave (or wave packet), we mean a real solution of the wave equation

$$(\partial_t^2 - \partial_{x_1}^2 - \ldots \partial_{x_d}^2 + m^2)\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes $\Phi$ by the stress-energy tensor $T_{\mu\nu}$, that provides the energy-momentum density of $\Phi$ at any time.

But, how to define the information, or entropy, carried by $\Phi$ in a given region at a given time?

*We give a classical answer to such a classical question by Operator Algebras and Quantum Field Theory*

# Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace. *Symplectic complement*:

$$H' = \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

$H$ is *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* if $H \cap iH = \{0\}$.

A standard subspace $H$ of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. $H$ is standard iff $H'$ is standard.

$H$ standard subspace $\rightarrow$ anti-linear operator $S : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}$,

$$S : \xi + i\eta \rightarrow \xi - i\eta, \ \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. $S$ is closed and densely defined, indeed

$$S_H^* = S_{H'}$$

# Modular theory for standard subspaces

Set $S = J\Delta^{1/2}$, polar decomposition of $S = S_H$.

Then $J$ is an anti-unitary involution, $\Delta > 0$ is non-singular and $J\Delta J = \Delta^{-1}$.

$$\Delta^{it} H = H, \quad JH = H'$$

(one particle Tomita-Takesaki theorem).

Eackmann, Osterwalder; Rieffel, van Daele; Leyland, Roberts, Testard

$\log \Delta$ is called the modular Hamiltonian in first quantization.

# Passivity

$\log \Delta_H$ is characterised (up to a proportionality constant) by complete passivity, following Pusz and Woronowicz.

$\mathcal{H}$ a complex Hilbert space, $H \subset \mathcal{H}$ a standard subspace and $A$ a selfadjoint linear operator on $\mathcal{H}$ such that $e^{isA}H = H$, $s \in \mathbb{R}$.

We shall say that $A$ is *passive* with respect to $H$ if

$$-(\xi, A\xi) \geq 0\,, \quad \xi \in D(A) \cap H\,.$$

$A$ is *n-passive* w.r.t. $H$ if the generator of $e^{itA} \otimes e^{itA} \cdots \otimes e^{itA}$ is passive with respect to the $n$-fold tensor product $H \otimes H \otimes \cdots \otimes H$, $A$ is *completely passive* if $A$ is $n$-passive for all $n \in \mathbb{N}$.

*$A$ is completely passive with respect to $H$ iff $\log \Delta_H = \lambda A$ for some $\lambda \geq 0$.*

# Entropy of a vector relative to a real linear subspace

Our analysis relies on the concept of *entropy $S_k^H$ of a vector $k$ in a Hilbert space $\mathcal{H}$ w.r.t. a closed real linear subspace $H$ of $\mathcal{H}$.*

The formula for the entropy $S_k^H$ is

$$S_k = \Im(k, P_H i \log \Delta_H \, k) = \Re(k, \mathcal{E}_H k)$$

Here $P_H$ is the crucial cutting projection $P_H : H + H' \to H$

$$P_H : h + h' \mapsto h$$

In terms of the modular Hamiltonian

$$P_H i|_H = -i \coth(\log \Delta_H /2)|_H$$

The entropy operator

$$\mathcal{E}_H = iP_H i \log \Delta_H$$

is real-linear and positive.

# Properties of the entropy of a vector

Some of the main properties of the entropy of a vector are:

- $S_k^H \geq 0$ or $S_k^H = +\infty$ (*positivity*);
- If $K \subset H$, then $S_k^K \leq S_k^H$ (*monotonicity*);
- If $k_n \to k$, then $S_k^H \leq \liminf_n S_{k_n}^H$ (*lower semicontinuity*);
- If $H_n \subset H$ is an increasing sequence with $\overline{\bigcup_n H_n} = H$, then $S_k^{H_n} \to S_k^H$ (*monotone continuity*);
- If $k \in D(\log \Delta_H)$ then $S_k^H < \infty$ (*finiteness on smooth vectors*).

# Tomita-Takesaki modular theory

$\mathcal{M}$ a von Neumann algebra on the complex Hilbert space $\mathcal{H}$, namely $\mathcal{M} \subset B(\mathcal{H})$ is a $^*$algebra and $\mathcal{M}'' = \mathcal{M}$ ($\mathcal{M}$ is the commutant of the commutant of $\mathcal{M}$).

$\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$, $\Omega \in \mathcal{H}$. Set

$$H \equiv \overline{\mathcal{M}_{\rm sa}\Omega}$$

$\Delta = \Delta_H$ and $J = J_H$ modular operator and conjugation

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta^{it} X \Delta^{-it}$$

modular automorphisms **intrinsic dynamics associated with $\varphi$!**

$$J\mathcal{M}J = \mathcal{M}' \quad \text{on } \mathcal{H}$$

The generator of the modular operator unitary group $\Delta_{\varphi}^{it}$ is called the modular Hamiltonian $\log \Delta_{\varphi}$

# Quantum Field Theory. Local nets of von Neumann algebras

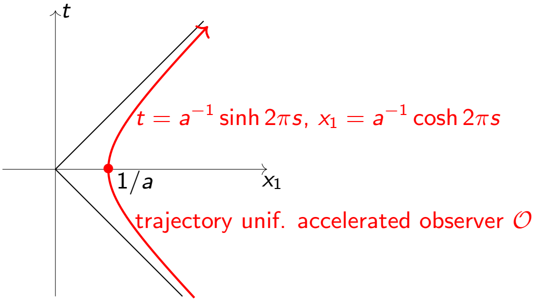
Haag-Kastler nets.

Local net $\mathcal{A}$ on spacetime $M$: map $O \subset M \mapsto \mathcal{A}(O) \subset B(\mathcal{H})$ s.t.

- *Isotony*, $O_1 \subset O_2 \implies \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- *Locality*, $O_1$, $O_2$ spacelike $\implies [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$
- *Poincaré covariance* (conformal, diffeomorphism) .
- *Positive energy and vacuum vector*.

# Bisognano-Wichmann theorem '75, Sewell comment '80

Rindler spacetime (wedge $x_1 > |t|$), vacuum modular group




$a$ : uniform acceleration of $O$

$s/a$ : proper time of $O$

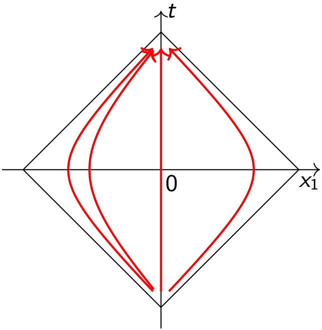
$\beta = 2\pi/a$ : inverse KMS temperature of $O$

Hawking-Unruh effect!

Time is geodesic, quantum gravitational effect!

# Double cone, conformal case

For a bounded region $O$ (double cone, causal envelop of a space ball $B$), in the conformal case the modular group is given by the geometric transformation (Hislop, L. '81)




local modular trajectories

$$(u, v) \mapsto \big((Z(u,s), Z(v,s)\big)$$

$$Z(z,s) = \frac{(1+z)+e^{-s}(1-z)}{(1+z)-e^{-s}(1-z)}$$

$$u = x_0 + r, \quad v = x_0 - r, \quad r = |\mathbf{x}| \equiv \sqrt{x_1^2 + \cdots + x_d^2}$$

# Entropy of finite systems

$X = \{x_1, \dots x_n\}$ a set of events. If $x_i$ occurs with probability $p_i$, its information is $-\log p_i$

$$\text{Shannon entropy} : S(P) = -\sum p_i \log p_i \ .$$

If $Q = \{q_1, \dots q_n\}$ other probability distribution (state)

$$\text{Relative entropy} : S(P\|Q) = \sum p_i(\log p_i - \log q_i)$$

mean value in the state $P$ of the difference between the information carried by the state $P$ and the state $Q$.

*Noncommutative entropy:* $\varphi = -\mathrm{Tr}(\rho_\varphi \,\cdot)$ state on a matrix algebra

$$\text{von Neumann entropy} : S(\varphi) = \ -\mathrm{Tr}(\rho_\varphi \log \rho_\varphi)$$

Umegaki's relative entropy

$$S(\varphi\|\psi) =: \mathrm{Tr}\big(\rho_\varphi(\log \rho_\varphi - \log \rho_\psi)\big)$$

# Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type $I$ so Tr does not exist; however Araki's relative entropy between two faithful normal states $\varphi$ and $\psi$ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where $\xi, \eta$ are cyclic vector representatives of $\varphi, \psi$ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with $\xi, \eta$.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

$$S(\varphi) = \sup_{(\varphi_i)} \sum_i \lambda_i S(\varphi_i \| \varphi)$$

supremum over all finite convex decompositions $\varphi = \sum_i \lambda_i \varphi_i$

# Weyl algebra and Gaussian states

If $H$ is a real linear space with a non-degenerate symplectic form $\beta$, The Weyl $C^*$-algebra $C^*(H)$ linearly generated by the (unitaries) $V(h)$, $h \in H$, that satisfy the commutation relations (CCR)

$$V(h+k) = e^{i\beta(h,k)} V(h)V(k)\,, \quad h, k \in H\,,$$

$V(h)^* = V(-h)$. A state $\varphi_\alpha$ on $C^*(H)$ is called Gaussian, if

$$\varphi_\alpha\big(V(h)\big) = e^{-\frac{1}{2}\alpha(h,h)}\,,$$

with $\alpha$ a real bilinear form $\alpha$ on $H$, compatible with $\beta$.

# Entropy of coherent sectors

$H$ standard linear subspace of $\mathcal{H}$ $\to$ von Neumann
$\mathcal{A}(H) = C^*(H)''$ algebra on $e^{\mathcal{H}}$
Given $\Phi \in \mathcal{H}$ consider the automorphism of $\mathcal{A}(H)$

$$\beta_\Phi = \mathrm{Ad}V(\Phi)^*|_{\mathcal{A}(H)} \ .$$

The vacuum relative entropy of the Gaussian state $\omega \cdot \beta_\Phi$ on $R(H)$ is given by the entropy of the vector $\Phi$ w.r.t. $H$. Namely by

$$S(\omega \cdot \beta_\Phi \| \omega) = S_\Phi$$

Araki's relative entropy

Entropy of vector

*Entropy of coherent state*

*Entropy of wave packet*

# First and second quantisation

*First quantisation*: map

$$O \subset \mathbb{R}^d \mapsto H(O) \text{ real linear space of } \mathcal{H}$$

local, covariant, etc.

*Second quantisation*: map

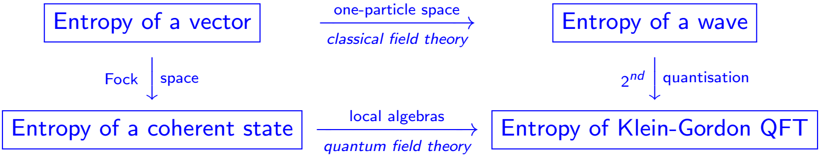
$$O \subset \mathbb{R}^d \mapsto \mathcal{A}(O) \equiv \mathcal{A}(H(O)) \text{ v.N. algebra on } e^{\mathcal{H}}$$

In our case $H(O)$ is generated by the waves with Cauchy data in $B$ ($O$ double cone with time-zero basis $B$)

*The core of our analysis in this talk will be done in first quantisation, then we apply the second quantisation functor.*

# Classical waves and quantum particle waves

The logical dependence in our construction is the following




The entropy of a vector $\Phi$ with respect to a real linear subspace $H(O)$ has different interpretations: classically, it measures the information carried by a wave packet in the spacetime region $O$; from the quantum point of view, it gives the vacuum relative entropy, on the observable algebra $\mathcal{A}(O)$ of the coherent state induced by $\Phi$ on the Fock space.

Indeed the same mathematical object $\Phi$ can describe:

- A classical wave packet
- The wave function of a free quantum particle
- A coherent state in QFT

# Entropy of a wave

Waves' time-independent symplectic form

$$\beta(\Phi, \Psi) = \frac{1}{2}\int_{x^0=t} (\Phi'\Psi - \Psi'\Phi) dx \ ,$$

The symplectic form is the imaginary part of complex Hilbert space scalar product (that depends on the mass).

Waves with Cauchy data supported in region $O$ (causal envelop of a space region $B$) form a real linear subspace $H(O) \equiv H(B)$.

The information carried by the wave $\Phi$ in the region $O$ is the entropy $S_\Phi$ of the vector $\Phi$ w.r.t. $H(O)$

By the Bisognano-Wichmann theorem, we know the modular group of $H(W)$, with $W$ a wedge ($B$ the half-space $x^1 \geq 0$)

The cutting projection cuts the Cauchy data: $f, g \mapsto \chi_B f, \chi_B g$.

# (Non)-positivity of local energy in QFT

In Quantum Field Theory local energy cannot be positive: If $f \geq 0$ is a smooth function with compact support then

$$T_{00}(f) \equiv \int f(x) T_{00}(x) dx \ngeq 0$$

(Epstein, Glaser, Jaffe '65).

Positivity of the energy is a fundamental physical aspect; it is essentially a manifestation of the stability of matter and may be expressed in different forms. In relativistic classical field theories of gravitation, in particular General Relativity, the Null Energy Condition plays an important role, for example in the work of Hawking and Penrose on spacetime singularities

# Energy conditions in QFT

In QFT, Bousso, Fisher, Liechenauer, and Wall proposed the Quantum Null Energy Condition, QNEC: For null direction deformations

$$\langle T_{uu} \rangle \geq \frac{1}{2\pi} S''_A(\lambda) \ ,$$

here $T$ stress-energy tensor, $S_A$ entanglement entropy relative to the deformed region $A$, $S''_A$ second derivative of $S_A$ w.r.t. the deformation parameter $\lambda$.

Physical arguments give the QNEC from the inequality

$$S''(\lambda) \geq 0$$

with $S(\lambda)$ Araki's relative entropy of every state w.r.t. the vacuum.

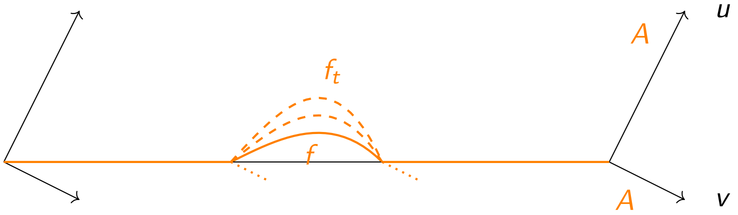



Figure: The function $f$ is the boundary of the deformed region on the null horizon. The entire deformed region is its causal envelop $A$.

Positivity of the second derivative of the *relative entropy* appears unexpectedly: $S''(\lambda) \geq 0$

See Ceyhan and Faulkner '19.

# The decomposition by Morinelli, Tanimoto and Wegner

In the free field, the modular group of a deformed wedge is geometric, it acts boostwise on each fiber.

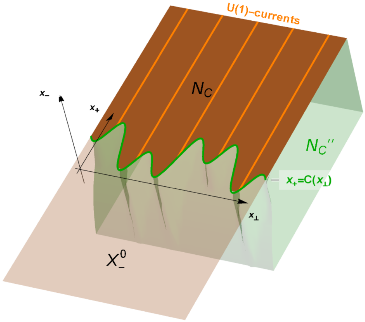



Figure: The fiber decomposition of the modular group

This result complements the Ceyhan and Faulkner result on QNEC for a dense subspace. General QNEC is open problem

# Entropy variation

Let $\Phi$ be a real Klein-Gordon wave and $H = H(W_\lambda)$. $W_\lambda$ null translated wedge.

The entropy $S_\Phi(\lambda)$ of $\Phi$ w.r.t. the wedge region $W_\lambda$ is the entropy of the vector $\Phi$ w.r.t. the standard subspace $H(W_\lambda)$.

$$S_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1\geq\lambda} (x^1 - \lambda)\, T_{00}(x)\, dx$$

then

$$S''_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1=\lambda} \langle v, Tv\rangle dx \geq 0 \ ,$$

where $v$ is the light-like vector $v = (1, 1, 0 \ldots, 0)$ *QNEC for coherent states, constant deformations*. (Work with F. Ciolli, G. Ruzzi.)

Here the energy density is

$$\langle T_{00}\rangle_\Phi = \frac{1}{2}\big(\Phi'^2 + |\nabla\Phi|^2 + m^2\Phi^2\big)$$

# Entropy and Klein–Gordon field on a globally hyperbolic spacetime

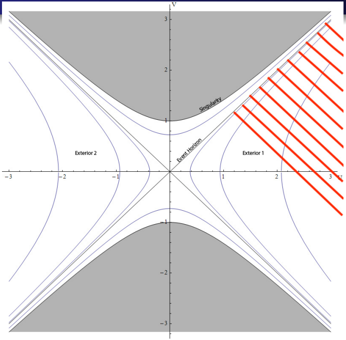



Figure: Schwarzschild-Kruskal spacetime. The red area is a null translated wedge

The convexity of the entropy w.r.t. to the null translation parameter has been obtained very recently for a Klein–Gordon field on a globally hyperbolic spacetime for coherent states (Ciolli, Ranallo, Ruzzi, L.). (Hollands, Ishibashi untraslated case)

# A basic abstract result

Let $H \subset \mathcal{H}$ be a standard subspace and $T(s) = e^{iAs}$ a one-parameter unitary group on $\mathcal{H}$ such that (Borchers pair)

- $A \geq 0$
- $T(s)H \subset H$, $s \geq 0$

Define $H_\lambda = T(\lambda)H\,,\ \lambda \in \mathbb{R}$, translated subspaces. Then the entropy function

$$\lambda \mapsto S(\lambda) = S_\psi^{H_\lambda} = \Im(\psi, P_{H_\lambda} i \log \Delta_{H_\lambda} \psi) \text{ is convex for all } \psi$$

and finite for a dense set of vectors. If $S(\lambda_0) < \infty$, then

(i) $S(\lambda)$ is finite and $C^1$ on $[\lambda_0, \infty)$;

(ii) $S'(\lambda)$ is absolutely continuous in $[\lambda_0, \infty)$ with almost everywhere non-negative derivative $S''(\lambda) \geq 0$ .

More general results are due to H. Bostelmann, D. Cadamuro, S. Del Vecchio

# The formula for the local massless Hamiltonian

The modular Hamiltonian $\log \Delta_B$ associated with the unit ball $B$ in the free scalar, massless QFT is (on Cauchy data)

$$-2\pi A = \log \Delta_B \, .$$

$$\log \Delta_B = \pi \imath \begin{bmatrix} 0 & (1-r^2) \\ (1-r^2)\nabla^2 - 2r\partial_r - D & 0 \end{bmatrix}$$

the higher dimensional Legendre operator

$$L = (1-r^2)\nabla^2 - r\partial_r$$

(Work with G. Morsella)

Massive case: interesting numerical results by H. Bostelmann, D. Cadamuro, C. Minz

# Local information in a wave packet

With $S_\Phi(R)$ the entropy of $\Phi$ in the radius $R$ ball cantered at $\bar{\mathbf{x}}$, we have

$$S_\Phi(R) = \pi \int_{B_R(\bar{\mathbf{x}})} \frac{R^2 - r^2}{R} \langle T_{00}(t, \mathbf{x}) \rangle_\Phi d\mathbf{x} \quad \text{stress-energy tensor term}$$
$$+ \pi \frac{d-1}{2R} \int_{B_R(\bar{\mathbf{x}})} \Phi^2(t, \mathbf{x}) d\mathbf{x} \quad \text{Born type term}$$

with $r = |\mathbf{x} - \bar{\mathbf{x}}|$
Here the energy density is

$$\langle T_{00} \rangle_\Phi = \frac{1}{2}\left(\Phi'^2 + |\nabla\Phi|^2\right)$$

# The parabolic distribution

$$S_\Phi(R) = \pi \int_{B_R(\bar{\mathbf{x}})} \underbrace{\frac{R^2 - r^2}{R}}_{\textit{parab. distr.}} \langle T_{00}(t,\mathbf{x})\rangle_\Phi d\mathbf{x} + \cdots$$

The parabolic distribution is a higher-dimensional generalization of Wigner semi-circular distribution in three-dimensional space (the marginal distribution function of a spherical distribution)

As $R \to 0$

$$S_\Phi(R,\mathbf{x}) = \frac{\pi}{d}\left(\langle T_{00}\rangle_\Phi(t,\mathbf{x}) + D\Phi^2(t,\mathbf{x})\right) A_{d-1}(R) + \ldots$$

Here $A_{d-1}(R) = 2\frac{\pi^{d/2}}{\Gamma(d/2)}R^{d-1}$ is the area of boundary sphere $\partial B_R$, cf. holographic area theorems, black holes (Bekenstein) and other contexts.

# Signal communication

Communication engineers consider signals, that is real functions, $r(t)$, on the time real line that are square-integrable

$$\text{Energy} \; = ||r||_2^2 = \int_{-\infty}^{\infty} |r(t)|^2 dt < \infty$$

e.g. $r(t)$ represents the voltage difference at time $t$ between two points in an electrical circuit. The Fourier transform of $r$ is the *amplitude spectrum* $R$

$$R(s) = \int_{-\infty}^{\infty} r(t) e^{-2\pi ts} dt$$

Note that $||r||_2 = ||R||_2$ by Plancherel theorem.

# Band-limited signals

Because of the limit of the devices, one has to consider band-limited signals, whose amplitude spectra vanish is $s > W$. Here $W > 0$ is a fixed parameter. Similarly, r(t) is said to be time-limited if for some $T > 0$, $r(t)$ vanishes for all $|t| > T/2$

$$R(s) = \int_{-W}^{W} r(t)e^{-2\pi ts}dt$$

As is well-known, the only signal that is both band-limited and time-limited is the trivial always-zero signal.

# Prolate operator

In signal communication an effective signal is cut in time and frequency. One has to maximize simultaneously

$$||f||_{2,B}/||f||_2 \quad \text{and} \quad ||\hat{f}||_{2,B}/||\hat{f}||_2$$

A famous work by Slepian et al. in the 70's considers the eigenvalue problem for the truncated Fourier transform, namely the positive Hilbert-Schmidt $T_B$ on $L^2(\mathbb{R}^d)$ operator with kernel $k_B(x-y)$ where

$$k_B(z) = \frac{1}{(2\pi)^d} \int_B e^{-ix\cdot z} dx \, \chi_B(z) \, .$$

The first eigenvalues are very close to 1.

The lucky accident was that this operator commutes with a linear differential operator, the prolate operator.

The prolate operator is

$$W = \nabla \cdot (1 - r^2)\nabla - r^2 = L$$

The prolate operator was recently considered by A. Connes ($d = 1$), in relation to the zeta functions. He $W$ is Hermitian, but not selfadjoint on $S(\mathbb{R}^d)$, it admits a unique selfadjoint extension on $L^2(\mathbb{R}^d)$ that commutes with $E_B$ and $\hat{E}_B$, where $E_B$ is the orthogonal projection from $L^2(\mathbb{R}^d)$ to $L^2(B)$, e similarly for $\hat{E}_B$ in Fourier transform.

# Entropy and prolate operator

$L$ the $d$-dimensional Legendre operator, acting on $\mathcal{S}$, defined by

$$L = \nabla(1 - r^2)\nabla\,,$$

namely $L$ is the divergence of the vector field $\frac{1}{2}(1 - r^2)\nabla$, $L$ can be written as

$$L = (1 - r^2)\nabla^2 - 2r\partial_r\,,$$

$L_D = L - D$, $D = d - 1$.

On $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, the entropy operator is

$$\mathcal{E} = \chi_B \cdot (-L_D) \oplus \chi_B \cdot M = \chi_B \cdot (-W_D + M) \oplus \chi_B \cdot M$$

with $M = r^2$,

$$W_D = W - D = L_D - r^2$$

if $\Phi$ is the wave with Cauchy data $f, g$, we have

$$S_\Phi = \pi\Big( -(f, W_D f)_B + (f, Mf)_B + (g, Mg)_B \Big)$$

sum of the "prolate entropy" and the "Born probabilities" with parabolic distributions.

As a consequence, we have the inequalities

$$-\pi(f, W_D f)_B \le S_\Phi \le -\pi(f, W_D f)_B + \pi||f||_B^2 + \pi||g||_B^2 \, .$$

The prolate operator is related to the entropy operator.

# Classification of factors

Murray-von Neumann:

A factor M, i.e. a von Neumann algebra with trivial center, can be of three types:

- Type $I_n$, $n = 1, 2, \ldots \infty$: $M \simeq B(\mathcal{H}) \otimes 1$, $\dim H = n$
- Type $II_1$: $M$ has a tracial state, $\dim M = \infty$;
  Type $II_\infty$: $M$ has a normal tracial weight, no minimal proj.
  $II_\infty = II_1 \otimes I_\infty$
- Type $III$: no trace (every non-zero projection is equiv. to 1)

Alain Connes:

- Type $III$ factors are divided into type $III_\lambda$ factors, $\lambda \in [0, 1]$.

# Continuous entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type $I$ so Tr does not exist; in the type II case, there exists a trace $\tau$ such that $\tau(\textit{projection})$ can be any non-negative value ($\mathrm{II}_\infty$) or $[0,1]$ ($\mathrm{II}_1$). A state $\varphi$ is given by a density matrix

$$\varphi = \tau(\rho\,\cdot)$$

and the entropy was defined by I. E. Segal as

$$S_\tau(\varphi) = -\tau(\rho \log \rho)$$

The entropy increase for an inclusion $\mathcal{N} \subset \mathcal{M}$ is bounded by the logarithm of the Jones index

$$0 \leq S_\tau(\varphi|_{\mathcal{N}}) - S_\tau(\varphi) \leq \log[\mathcal{M} : \mathcal{N}]$$

the bound is optimal if $\mathcal{M}$ is infinite-dimensional, E. Witten and R. L. Feb. 2022

# Witten's states

$M$ a type $III_1$ factor on $\mathcal{H}$. The crossed product $\tilde{M}$ of $M$ by the modular group $\sigma^\varphi$ is the von Neumann algebra on $\mathcal{H} \otimes L^2(\mathbb{R})$ generated by

$$M \otimes 1 \quad \text{and} \quad \Delta_\varphi^{is} \otimes V(s)$$

with $V$ the translation group. By Connes-Takesaki duality, $\tilde{M}$ is a $II_\infty$ factor.

In relation to black hole physics, Witten considers vector states on $\tilde{M}$ of the form $\xi_\varphi \otimes g$, with $g$ a positive $L^2$ function and the $II_\infty$ entropy of these states.

# Type $II_1$ entropy and crossed product

Let $\mathcal{M}$ be a type $III_1$ factor acting standardly on $\mathcal{H}$. The Connes-Takesaki crossed product by the modular group $\sigma^\varphi$ is the $II_\infty$ factor $\mathcal{N}$ on $\mathcal{H} \otimes L^2(\mathbb{R})$ generated by

$$u_t = \Delta^{it} \otimes U_t$$

$U$ translation unitary on $L^2(\mathbb{R})$. In the crossed product:

$$\text{generator of } u = \log \Delta_\varphi + \text{one-particle Hamiltonian}$$

With $E$ the positive energy spectral projection of $u$

$$\mathrm{trace}(E) = 1 < \infty$$

so $E\mathcal{N}E$ is a type $II_1$ factor, this leads to Witten's finite entropy states interpretation. V. Chandrasekaran, G. Penington, E. Witten, R. L.